Bounded Chromatic Polynomials

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Abstract

We study the analogue of chromatic polynomials in the setting of bounded vertex colorings of graphs. For a graph G and a positive integer b, define the function $\chi(G, b, k)$ to be the number of vertex colorings of G using (at most) k colors, in which every color is assigned to at most b vertices. We show that $\chi(G, b, k)$, considered as a function of k, is a monic polynomial whose degree is the number of vertices of G, and call it the *b-bounded chromatic polynomial* of G. We then specialize to $b = 2$ and give a reduction formula which allows us express $\chi(G, 2, k)$ in terms of the 2-bounded chromatic polynomials of null graphs. In this way, we find formulas for the 2-bounded chromatic polynomials of path graphs, cycle graphs, complete graphs, and complete bipartite graphs. We also give identities for the 2-bounded chromatic polynomials of graph complements, disjoint unions of graphs, and joins of graphs. Finally, we give a simple combinatorial interpretation for the coefficients in our formulas.

1 Introduction

In this paper, we study the analogue of chromatic polynomials in the setting of bounded vertex colorings of graphs. Suppose that G is a graph and b a positive integer. We will say that a *b-bounded vertex coloring* of G is an ordinary vertex coloring in which each color is used at most b times. The earliest reference we have found to this variant of coloring is [7], although we would not be surprised to find that the problem is older. In [7], Hansen, Hertz, & Kuplinsky introduce this variant of coloring and study the problem of finding the *bounded chromatic number* $\gamma_b(G)$, the smallest number of colors such that a b-bounded coloring of G exists. Since then, a number of other papers ([1, 4, 6, 8]) have studied bounded colorings. (The study of graph colorings is extensive, and this is not intended to be a comprehensive list of references.) These papers are all concerned with either computing the bounded chromatic number of a graph, or else finding the computational complexity of this calculation. In particular, Bodlaender & Jansen [4] have shown that when $b \geq 3$, the problem of finding $\gamma_b(G)$ is NP-complete. Some extensions of this problem have also been discussed. Bandopadhyay et al. [3] studied a variant in which different colors may have different bounds on the number of times they can be used. In a different direction, Bampis et al. [2] studied a variant in which the vertices have weights and one seeks a coloring in which the sum of the maximum weight of each color is minimized. Again, these papers focus on the analogue of the chromatic number and the computational complexity of calculating it. As far as we can tell, the analogue of the chromatic polynomial in the bounded coloring setting has remained unstudied.

Given a graph G and number b, we define the *b-bounded chromatic polynomial*[1], $\chi(G, b, k)$, to be the function which assigns to each positive integer k the number of b-bounded colorings of G that use (at most) k colors. As with the usual chromatic polynomial, we will assume that G does not contain any loops or double edges. For some graphs and values of b, the value of $\chi(G, b, k)$ is easy to calculate. For example, if G is any graph with n vertices, then we trivially have

$$\chi(G, 1, k) = k(k-1)\cdots(k-n+1).$$

It is also easy to see that if K_n is the complete graph on n vertices, then for any b, we have

$$\chi(K_n, b, k) = \chi(K_n, 1, k) = k(k-1)\cdots(k-n+1) = \chi(K_n, k),$$

where $\chi(K_n, k)$ is the ordinary chromatic polynomial. From this point forward, we will use the customary notation

$$(k)_n = k(k-1)(k-2)\cdots(k-n+1)$$

for falling factorials, so that we have $\chi(G, 1, k) = \chi(K_n, b, k) = (k)_n$.

Our first goal in this paper will be to show that the function $\chi(G, b, k)$ is, in fact, a polynomial in k. We do this in the following theorem.

[1] It will be shown in Section 2 that this function really is a polynomial

Theorem 1. *Let G be a graph with n vertices and let b be a positive integer. Then $\chi(G, b, k)$ is a monic polynomial in k of degree n.*

Once we know that $\chi(G, b, k)$ is a polynomial in k, we turn our attention to computing this polynomial. For this, we almost exclusively restrict our attention to the case $b = 2$. It is shown in [4] that the problem of finding the b-bounded chromatic number of G is NP-complete when $b \geq 3$, and so it is unreasonable to expect that we will find explicit formulas for b-bounded chromatic polynomials in those cases. When $b = 2$, however, we are able to make some progress. We begin by finding explicit formulas for $\chi(N_n, 2, k)$, where N_n is the null graph on n vertices.

Theorem 2. *Let n be a positive integer, and for $j = 0, \ldots, n$, set $\Delta(j) = \min(\lfloor n/2 \rfloor, n - j)$. Then we have $\chi(N_n, 2, k) = \sum_{j=0}^{n} a_j k^j$, where*

$$a_j = \sum_{i=0}^{\Delta(j)} \frac{(-1)^{n-i-j} n!}{2^i i! (n-2i)!} \begin{bmatrix} n-i \\ j \end{bmatrix},$$

for each j, where $\begin{bmatrix} n-i \\ j \end{bmatrix}$ represents the unsigned Stirling number of the first kind.

In Table 1 and Table 2, we give the polynomials $\chi(N_n, 2, k)$ for $1 \leq n \leq 10$ and their factorizations into polynomials irreducible over the integers. We note that it seems to be difficult at best to find a simpler formula for $\chi(N_n, 2, k)$, or a simpler expression for the coefficients. It is not hard to see that $k - i$ is a factor of $\chi(N_n, 2, k)$ for all integers i with $0 \leq i < n/2$, but it seems to be difficult to see any pattern in the remaining factors.

After dealing with null graphs, we begin the study of $\chi(G, 2, k)$ in general. We begin by proving a reduction theorem that allows us to relate the b-bounded chromatic polynomial of G to the b-bounded chromatic polynomials of "smaller" graphs. This is reminiscent of the deletion-contraction theorem used for the ordinary chromatic polynomial.

Theorem 3. *Let G be a graph and let $e = \{v, w\}$ be an edge of G. Then we have*

$$\chi(G, 2, k) = \chi(G - e, 2, k) - k\chi(G - v - w, 2, k - 1). \quad (1)$$

Here, the graph $G - e$ is obtained from G by deleting the edge e, and the graph $G - v - w$ is obtained from G by deleting the vertices v and w, as well

n	$\chi(N_n, 2, k)$
1	k
2	k^2
3	$k^3 - k$
4	$k^4 - 4k^2 + 3k$
5	$k^5 - 10k^3 + 15k^2 - 6k$
6	$k^6 - 20k^4 + 45k^3 - 26k^2$
7	$k^7 - 35k^5 + 105k^4 - 56k^3 - 105k^2 + 90k$
8	$k^8 - 56k^6 + 210k^5 - 56k^4 - 840k^3 + 1371k^2 - 630k$
9	$k^9 - 84k^7 + 378k^6 + 84k^5 - 3780k^4 + 8819k^3 - 7938k^2 + 2520k$
10	$k^{10} - 120k^8 + 630k^7 + 588k^6 - 12600k^5 + 37295k^4 - 44730k^3 + 18936k^2$

Table 1: Polynomials $\chi(N_n, 2, k)$ for $1 \leq n \leq 10$.

as all edges attached to them. Equivalently, suppose that G is a graph and that v, w are vertices of G that are not connected by an edge. Write $G + e$ for the graph obtained by adding the edge $e = \{v, w\}$ to G. Then we have

$$\chi(G, 2, k) = \chi(G + e, 2, k) + k\chi(G - v - w, 2, k - 1). \tag{2}$$

Once we have this theorem, we can compute $\chi(G, 2, k)$ for any graph G by using Theorem 3 repeatedly until every remaining graph is a null graph, and then appealing to Theorem 2. Thus we may use the polynomials $\chi(N_n, 2, k)$ as a sort of basis for the 2-bounded chromatic polynomials. We prove this in the following theorem.

Theorem 4. *For any graph G with n vertices, there exists a unique sequence a_i of non-negative integers such that*

$$\chi(G, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k - i). \tag{3}$$

The word "basis" here is a bit of a misnomer, as not every set of integers a_i yields a 2-bounded chromatic polynomial. We find that for specific types of graphs, it is significantly easier to find formulas for the numbers $a_0, \ldots, a_{\lfloor n/2 \rfloor}$ than to find formulas for the coefficients of the monomials k^j.

After proving Theorem 4, we use Theorem 3 to find representations of $\chi(G, 2, k)$ in the form given in (3) for several different types of graphs. We

n	Factorization of $\chi(N_n, 2, k)$
1	k
2	k^2
3	$k(k-1)(k+1)$
4	$k(k-1)(k^2+k-3)$
5	$k(k-1)(k-2)(k^2+3k-3)$
6	$k^2(k-1)(k-2)(k^2-3k-13)$
7	$k(k-1)(k-2)(k-3)(k+1)(k^2+5k-15)$
8	$k(k-1)(k-2)(k-3)(k^4+6k^3-31k^2-36k+105)$
9	$k(k-1)(k-2)(k-3)(k-4)(k^4+10k^3-19k^2-112k+105)$
10	$k^2(k-1)(k-2)(k-3)(k-4)(k^4+10k^3-55k^2-220k+789)$

Table 2: Factorization of $\chi(N_n, 2, k)$ for $1 \leq n \leq 10$.

find formulas when G is a path graph, a cycle graph, a complete graph, or a complete bipartite graph. We also give interesting identities for the 2-bounded chromatic polynomials of graph complements, disjoint unions of graphs, and joins of graphs.

Finally, we give an interesting combinatorial interpretation for the numbers a_i in Theorem 4. This is similar in spirit to the combinatorial interpretation of the coefficients of k^j in the ordinary chromatic polynomial (see for example [9]). We show the following theorem.

Theorem 5. *Let G be a graph, and write $\chi(G, 2, k)$ as in Theorem 4. Then for each i, the number a_i equals the number of ways to choose a set of i pairwise disjoint edges (i.e., no two edges in the set share a common vertex) in G.*

2 $\chi(G, b, k)$ is a polynomial

In this section, we prove Theorem 1, that for any graph G and any bound b, the function $\chi(G, b, k)$ is a monic polynomial in k of degree n, where n is the number of vertices of G.

We proceed by induction on n. If $n = 1$, then $G = N_1$, and we clearly have $\chi(N_1, b, k) = k$ regardless of the value of b. Now suppose that for some number $n \geq 2$ we know that if H is any graph with $m < n$ vertices then $\chi(H, b, k)$ is a monic polynomial of degree m. Suppose that G is a graph with n vertices and let v be a vertex of G. Suppose that v is assigned a color c, and note that there are k choices for this color. Once this color is assigned, for each i with $1 \le i \le b$, we count the number of colorings of G in which the color c is used exactly i times. For each i, there are at most $\binom{n-1}{i-1}$ ways to choose the other vertices that have color c. (Note that this is zero if $i > n$.) For each such choice, the number of ways to color the remaining $n-i$ vertices is given by $\chi(G^*, b, k-1)$, where G^* is the subgraph of G obtained by removing all the vertices that have been colored with color c and all edges attached to these vertices[2]. By induction, each of these functions $\chi(G^*, b, k-1)$ is a monic polynomial in k of degree $n-i$. Adding all the polynomials obtained from all the different i, we see that the number of colorings of G in which v has color c is a sum of finitely many (at most $\binom{n-1}{0} + \cdots + \binom{n-1}{b-1} \le 2^{n-1}$) monic polynomials of degree at most $n-1$, and so the sum is a polynomial of degree at most $n-1$. Multiplying by k (since there were k choices for the color c) shows that $\chi(G, b, k)$ is a polynomial of degree at most n.

To show that this polynomial is monic and has degree exactly n, we note that when $i = 1$, the only subgraph that needs to be considered is $G^* = G - v$. Thus, when we sum all of the polynomials $\chi(G^*, b, k-1)$, only $\chi(G-v, b, k-1)$ has degree $n-1$, while the other summands all have smaller degrees. Hence the sum is monic of degree exactly $n-1$, and when multiplied by k becomes monic of degree n. This completes the proof of the theorem. □

3 Null Graphs

In this section, we find recursive relations for the functions $\chi(N_n, b, k)$, where N_n is the null graph on n vertices. We will then use the recursive nature of these functions to prove Theorem 2, finding a general formula for $\chi(N_n, 2, k)$. If specific values of n and b are given, then these relations, along with the fact that $\chi(N_1, b, k) = k$, are sufficient to determine the polynomial $\chi(N_n, b, k)$.

Lemma 6. *We have*

$$\chi(N_n, b, k) = k \sum_{i=1}^{b} \binom{n-1}{i-1} \chi(N_{n-i}, b, k-1). \tag{4}$$

[2] We acknowledge that we are slightly abusing notation here, since different choices of which vertices have color c lead to different subgraphs for G^*.

Proof. We prove this in the same way that we proved Theorem 1. Let v be one of the vertices of N_n. There are k ways to choose the color of v. Suppose that this color is c. For each value i with $1 \le i \le b$, we count the number of colorings of N_n in which v has color c and the color c is used exactly i times. After coloring v, there are exactly $\binom{n-1}{i-1}$ ways to choose which other vertices have color c. Once we make this choice, we need to color the remaining $n-i$ vertices with the remaining $k-1$ colors. There are $\chi(N_{n-i}, b, k-1)$ ways to do this. Thus, for each i, there are

$$\binom{n-1}{i-1}\chi(N_{n-i}, b, k-1)$$

colorings of N_n in which v has color c and this color is used exactly i times. To find $\chi(N_n, b, k)$, we sum this expression over all possible values $i = 1, \ldots, b$ to get the number of b-bounded colorings of N_n in which v has color c, and then multiply by k since there are k ways to choose the color c. This gives the formula in the statement of the lemma. □

Lemma 7. *We have*

$$\chi(N_n, b, k) = \chi(N_n, b-1, k) + \left(\sum_{i=1}^{\lfloor n/b \rfloor} \binom{k}{i} \prod_{j=0}^{i-1} \binom{n-jb}{b}\right) \chi(N_{n-ib}, b-1, k-i).$$

Proof. In a coloring of N_n, let i be the number of colors that are used exactly b times, and note that we have $0 \le i \le \lfloor n/b \rfloor$. If $i = 0$, then every color is used at most $b-1$ times, and there are $\chi(N_n, b-1, k)$ such colorings. Now, for a fixed positive i, we count the number of colorings of N_n in which exactly i colors are used b times. There are $\binom{k}{i}$ ways of picking which colors are used b times. Once the colors are chosen, call them $c_0, \ldots, c_{i-1}$. Then there are $\binom{n}{b}$ ways to choose which vertices have color c_0, and in general there are $\binom{n-jb}{b}$ ways to choose which vertices have color c_j. Once these vertices are chosen, we need to color the remaining $n-ib$ vertices. We have $k-i$ colors remaining since $c_0, \ldots, c_{i-1}$ have been used as many times as possible, and each of the remaining colors can be used at most $b-1$ times. Hence, for each i with $1 \le i \le \lfloor n/b \rfloor$ there are

$$\left(\binom{k}{i} \prod_{j=0}^{i-1} \binom{n-jb}{b}\right) \chi(N_{n-ib}, b-1, k-i)$$

colorings of N_n in which exactly i colors are used b times. To find $\chi(N_n, b, k)$, we sum over all possible values of i, which gives the formula in the statement of the lemma. □

Clearly, we can use Lemma 6 and Lemma 7 to explicitly calculate the polynomial $\chi(N_n, b, k)$ for any choice of n and b. In the case $b = 2$ we can in fact give a general formula for $\chi(N_n, 2, k)$. This is the content of Theorem 2, which we now prove.

Proof of Theorem 2. Using Lemma 7. We have

$$\chi(N_n, 2, k) = \chi(N_n, 1, k) + \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\binom{k}{i} \prod_{j=0}^{i-1} \binom{n-2j}{2} \right) \chi(N_{n-2i}, 1, k-i). \quad (5)$$

In the introduction, we saw that $\chi(N_n, 1, k) = (k)_n$. Next, noticing that

$$\prod_{j=0}^{i-1} \binom{n-2j}{2} = \prod_{j=0}^{i-1} \frac{(n-2j)(n-2j-1)}{2} = \frac{n!}{2^i (n-2i)!},$$

the expression (5) becomes

$$\begin{aligned} \chi(N_n, 2, k) &= (k)_n + \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{(k)_i}{i!} \cdot \frac{n!}{2^i (n-2i)!} \cdot (k-i)_{n-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i! (n-2i)!} \cdot (k)_{n-i}. \end{aligned} \quad (6)$$

We now wish to extract the coefficient of k^j from the expression (6). For this, let $\left[{n \atop j} \right]$ represent the (unsigned) Stirling number of the first kind. That is, $\left[{n \atop j} \right]$ gives the number of permutations of n elements that have j disjoint cycles. We now use the well-known formula (see for example [5, Equation (6.4)]) that

$$(k)_n = \sum_{j=0}^{n} (-1)^{n-j} \left[{n \atop j} \right] k^j.$$

Inserting this formula into (6), we obtain

$$\chi(N_n, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i!(n-2i)!} \sum_{j=0}^{n-i} (-1)^{n-i-j} \begin{bmatrix} n-i \\ j \end{bmatrix} k^j \tag{7}$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-i} \frac{(-1)^{n-i-j} n!}{2^i i!(n-2i)!} \begin{bmatrix} n-i \\ j \end{bmatrix} k^j. \tag{8}$$

We now interchange the order of summation. Note that for a given value of j, the value of i can be at most $\Delta(j) = \min(\lfloor n/2 \rfloor, n-j)$. This yields

$$\chi(N_n, 2, k) = \sum_{j=0}^{n} \sum_{i=0}^{\Delta(j)} \frac{(-1)^{n-i-j} n!}{2^i i!(n-2i)!} \begin{bmatrix} n-i \\ j \end{bmatrix} k^j,$$

which immediately gives the desired expression for a_j. □

4 Calculating 2-bounded chromatic polynomials

We now turn our attention to calculating 2-bounded chromatic polynomials for non-null graphs. We begin by proving Theorem 3, which gives a relation between $\chi(G, 2, k)$ and the 2-chromatic polynomials of "smaller" graphs.

Proof of Theorem 3. First, to see that the two formulas are equivalent, suppose that $e = \{v, w\}$ is an edge of G and let $H = G - e$. Then we see that $G = H + e$ and that $G - v - w$ and $H - v - w$ are the same graph. Hence either formula can be obtained from the other, proving equivalence.

Now we prove that the equation (2) holds. Suppose that the vertices v, w of G are not connected by an edge. In a 2-bounded coloring of G, these vertices may either have the same color or different colors. The 2-bounded colorings of G with v, w having different colors correspond exactly to the 2-bounded colorings of $G + e$. To count the number of 2-bounded colorings of G in which v and w have the same color, suppose that this common color is c. For each choice of c, we must count the number of ways to color the remaining vertices. Since c has been used twice, none of the remaining vertices can have color c, and therefore neither the vertices v and w, nor the edges emanating from them, have any influence on how the remaining vertices can be colored.

That is, the number of ways to color the remaining vertices is the same as the number of ways to color the graph $G - v - w$ using $k - 1$ colors, which is $\chi(G - v - w, 2, k - 1)$. Since there were k ways to choose the color c, the total number of 2-bounded colorings of G in which v, w have the same color is $k\chi(G - v - w, 2, k - 1)$. Therefore the total number of 2-bounded colorings of G is

$$\chi(G, 2, k) = \chi(G + e, 2, k) + k\chi(G - v - w, 2, k - 1),$$

as desired. This completes the proof of the theorem. □

We now embark on the proof of Theorem 4, showing how to express $\chi(G, 2, k)$ in terms of the 2-bounded chromatic polynomials of null graphs. To prove this theorem, we first need the following simple lemma and its corollary.

Lemma 8. *Let G be a graph with n vertices and let H be a graph that is disjoint from G. Suppose that by using Theorem 3 repeatedly, we can write $\chi(G, 2, k)$ as in (3), so that we have*

$$\chi(G, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k - i).$$

Then we have

$$\chi(G \cup H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n-2i} \cup H, 2, k - i). \tag{9}$$

Proof. Suppose that $e = \{v, w\}$ is an edge of G. Then by Theorem 3, we have

$$\begin{aligned} \chi(G \cup H, 2, k) &= \chi((G \cup H) - e, 2, k) - k\chi((G \cup H) - v - w, 2, k - 1) \\ &= \chi((G - e) \cup H, 2, k) - k\chi((G - v - w) \cup H, 2, k - 1). \end{aligned}$$

In this way, we see that in any formula for $\chi(G, 2, k)$ that can be obtained by repeated uses of Theorem 3, we may replace each subgraph G^* in the formula with $G^* \cup H$ to obtain a formula for $\chi(G \cup H, 2, k)$. Thus we can replace each graph N_{n-2i} in (3) by $N_{n-2i} \cup H$ to obtain (9). This completes the proof of the lemma. □

Corollary 9. *Suppose that G is a graph and that N_j is a null graph disjoint from G. If we can use Theorem 3 repeatedly to write $\chi(G, 2, k)$ as in* (3), *then we have*

$$\chi(G \cup N_j, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n+j-2i}, 2, k-i).$$

Proof. This is immediate from Lemma 8 and noting that if N_m and N_n are disjoint, then $N_m \cup N_n = N_{m+n}$. □

Proof of Theorem 4. We prove the theorem by induction on n. If $n = 1$, then we have $G = N_1$ and we can check that (3) holds with $a_0 = 1$. If $n = 2$, then either $G = N_2$, when we have a_0=1 and $a_1 = 0$, or G is the complete graph $G = K_2$, in which case we have $a_0 = a_1 = 1$. Suppose now that for some number $n \geq 3$, we know that the theorem is true for all graphs with $m < n$ vertices. Let G be a graph with n vertices. If it happens that $G = N_n$, then we may clearly take $a_0 = 1$ and $a_i = 0$ for $1 \leq i \leq \lfloor n/2 \rfloor$. Suppose then that G has edges and choose a vertex v of G that is attached to at least one edge. Suppose that v has t edges, $\{v, w_1\}, \ldots, \{v, w_t\}$. Using the relation (1) t times, we may successively remove all the edges attached to v. This yields

$$\chi(G, 2, k) = \chi((G - v) \cup N_1, 2, k) - k \sum_{j=1}^{t} \chi(G - v - w_j, 2, k-1).$$

Now, the graph $G - v$ has $n - 1$ vertices and so by the inductive hypothesis $\chi(G - v, 2, k)$ can be written in the form (3). Hence Corollary 9 implies that there are integers g_i such that

$$\chi((G - v) \cup N_1, 2, k) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i g_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i). \tag{10}$$

Also, since each graph $G - v - w_j$ has $n - 2$ vertices, the inductive hypothesis applies to each of these graphs as well. Thus, for each graph $G - v - w_j$, there exist positive integers $a_i(j)$ such that

$$\begin{aligned}
&- k\chi(G - v - w_j, 2, k-1) \\
&\qquad = -k \sum_{i=0}^{\lfloor n/2 \rfloor - 1} (-1)^i a_i(j) \cdot (k-1)_i \cdot \chi(N_{n-2-2i}, 2, k-1-i) \\
&\qquad = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} (-1)^{i+1} a_i(j) \cdot (k)_{i+1} \cdot \chi(N_{n-2(i+1)}, 2, k-(i+1)) \\
&\qquad = \sum_{u=1}^{\lfloor n/2 \rfloor} (-1)^u a_{u-1}(j) \cdot (k)_u \cdot \chi(N_{n-2u}, 2, k-u),
\end{aligned}$$

where we have set $u = i+1$ in the last expression. Summing the expressions for the $-k\chi(G - v - w_j, 2, k-1)$, we see that there are integers h_i such that

$$-k \sum_{j=1}^{t} \chi(G - v - w_j, 2, k-1) = \sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i h_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i). \quad (11)$$

Adding (10) and (11), we can see that $\chi(G, 2, k)$ can be written in the form given in the statement of the theorem. This completes the existence part of the proof.

In order to prove that the numbers a_i are unique, suppose that we have two representations of $\chi(G, 2, k)$ in the form (3) with coefficients a_i and b_i. Then we have

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i b_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i),$$

which immediately gives

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i (a_i - b_i) \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) = 0. \quad (12)$$

Now, write $f_i(k) = (k)_i \cdot \chi(N_{n-2i}, 2, k-i)$. We can see that for each i, the polynomial f_i has degree $n-i$. Since f_0 is the only summand on the left-hand side of (12) containing a term of degree n, the only way the left-hand side can be identically zero is to have $a_0 = b_0$. Since $a_0 - b_0 = 0$, we then have

$$\sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i (a_i - b_i) \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) = 0.$$

Proceeding in the same way, we can see that $a_1 = b_1$, and in fact that we must have $a_i = b_i$ for all i. This completes the uniqueness part of the proof. $\square$

5 2-bounded chromatic polynomials for special graphs

In this section, we find formulas for the 2-bounded chromatic polynomial $\chi(G, 2, k)$ for certain types of graphs. By "formula" in this context, we mean that we find the coefficients a_i in Theorem 4.

5.1 Paths

Theorem 10. *Suppose that P_n is the path graph with n vertices. Then we have*

$$\chi(P_n, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i). \tag{13}$$

Proof. We prove this by induction on n. For $n = 1, 2, 3$, we can see by direct calculation that (13) holds. Now let $m \geq 1$ be an integer and suppose we know that (13) holds for all $n \leq 2m+1$. If $n = 2m+2$, we use Theorem 3 to delete a terminal edge, obtaining

$$\chi(P_{2m+2}, 2, k) = \chi(P_{2m+1} \cup N_1, 2, k) - k\chi(P_{2m}, 2, k-1).$$

From the inductive hypothesis and Corollary 9, we see that we have

$$\begin{aligned}\chi(P_{2m+2}, 2, k) = \sum_{i=0}^{m} (-1)^i \binom{2m+1-i}{2m+1-2i} \cdot (k)_i \cdot \chi(N_{2m+2-2i}, 2, k-i) \\ - k \sum_{i=0}^{m} (-1)^i \binom{2m-i}{2m-2i} \cdot (k-1)_i \cdot \chi(N_{2m-2i}, 2, k-1-i).\end{aligned}$$

By separating the $i = 0$ term from the first sum and reindexing the second sum so that it starts at $i = 1$, we obtain

$$\chi(P_{2m+2}, 2, k) = \sum_{i=1}^{m} (-1)^i \binom{2m+1-i}{2m+1-2i} \cdot (k)_i \cdot \chi(N_{2m+2-2i}, 2, k-i)$$
$$+ \sum_{i=1}^{m} (-1)^i \binom{2m+1-i}{2m+2-2i} \cdot (k)_i \cdot \chi(N_{2m+2-2i}, 2, k-i)$$
$$+ \chi(N_{2m+2}, 2, k) + (-1)^{m+1} \cdot (k)_{m+1} \cdot \chi(N_0, 2, k-m-1).$$

Combining the two summations above, we see that $\chi(P_{2m+2}, 2, k)$ equals

$$\chi(N_{2m+2}, 2, k) + (-1)^{m+1} \cdot (k)_{m+1} \cdot \chi(N_0, 2, k-m-1)$$
$$+ \sum_{i=1}^{m} (-1)^i \binom{2m+2-i}{2m+2-2i} \cdot (k)_i \cdot \chi(N_{2m+2-2i}, 2, k-i).$$

Finally, we combine all these terms into a single summation, obtaining

$$\chi(P_{2m+2}, 2, k) = \sum_{i=0}^{m+1} (-1)^i \binom{2m+2-i}{2m+2-2i} \cdot (k)_i \cdot \chi(N_{2m+2-2i}, 2, k-i),$$

as desired.

We treat P_{2m+3} similarly. Using Theorem 3, we can express $\chi(P_{2m+3}, 2, k)$ as

$$\chi(P_{2m+2} \cup N_1, 2, k) - k\chi(P_{2m+1}, 2, k-1).$$

Using the induction hypothesis and the fact that the theorem is true for P_{2m+2}, this expression equals

$$\sum_{i=0}^{m+1} (-1)^i \binom{2m+2-i}{2m+2-2i} \cdot (k)_i \cdot \chi(N_{2m+3-2i}, 2, k-i)$$
$$- k \sum_{i=0}^{m} (-1)^i \binom{2m+1-i}{2m+1-2i} \cdot (k-1)_i \cdot \chi(N_{2m+1-2i}, 2, k-1-i).$$

Following essentially the same algebraic steps as before yields the formula

$$\chi(P_{2m+3}, 2, k) = \sum_{i=0}^{m+1} (-1)^i \binom{2m+3-i}{2m+3-2i} \cdot (k)_i \cdot \chi(N_{2m+3-2i}, 2, k-i),$$

as desired. We have now shown that the theorem is true for P_n whenever $n \leq 2(m+1)+1$, completing the induction. This completes the proof of the theorem. □

5.2 Cycles

Theorem 11. *Let C_n be the cycle graph with n vertices. Then we have*

$$\chi(C_n, 2, k) = \chi(N_n, 2, k) + \sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i \frac{n}{i} \binom{n-i-1}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i).$$

Proof. By Theorem 3, we have

$$\chi(C_n, 2, k) = \chi(P_n, 2, k) - k\chi(P_{n-2}, 2, k-1).$$

Using (13), we see that $\chi(C_n, 2, k)$ equals

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) \\ - k \sum_{i=0}^{\lfloor n/2 \rfloor - 1} (-1)^i \binom{n-2-i}{n-2-2i} \cdot (k-1)_i \cdot \chi(N_{n-2-2i}, 2, k-1-i).$$

Separating the $i = 0$ term from the first sum and reindexing the second sum to start at $i = 1$ shows that $\chi(C_n, 2, k)$ equals

$$\chi(N_n, 2, k) \\ + \sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) \\ + \sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-1-i}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i).$$

Finally, combining the two summations shows that $\chi(C_n, 2, k)$ equals

$$\chi(N_n, 2, k) + \sum_{i=1}^{\lfloor n/2 \rfloor} (-1)^i \frac{n}{i} \binom{n-i-1}{n-2i} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i),$$

as desired. This completes the proof of the theorem. □

5.3 Graph Complements and Complete Graphs

In this section, we find a relationship between the 2-bounded chromatic polynomials of a graph and its complement. In this formula, noting that the complement of N_n is K_n, we express $\chi(G^c, 2, k)$ in terms of the 2-bounded chromatic polynomials of complete graphs.

Theorem 12. *Let G be a graph with n vertices and let G^c be its complement. If we have*

$$\chi(G, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i),$$

then

$$\chi(G^c, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i \cdot (k)_i \cdot \chi(K_{n-2i}, 2, k-i). \tag{14}$$

Proof. We proceed by induction on n. The cases with $n \leq 2$ are trivial. Now suppose that for some $V \geq 3$, the theorem is true for any graph G with $n < V$ vertices. Let G be a graph with V vertices. Note that when G has no edges, $G = N_V$, and the theorem is clearly true with $a_0 = 1$ and $a_1 = \cdots = a_{\lfloor n/2 \rfloor} = 0$. Now suppose that for some number $M \geq 1$, the theorem is true for any graph G with V vertices and $m < M$ edges. Suppose G has V vertices and M edges. Let $e = \{v, w\}$ be an edge of G and suppose that

$$\chi(G - e, 2, k) = \sum_{i=0}^{\lfloor V/2 \rfloor} (-1)^i b_i \cdot (k)_i \cdot \chi(N_{V-2i}, 2, k-i)$$

and

$$\chi(G - v - w, 2, k) = \sum_{i=0}^{\lfloor V/2 \rfloor - 1} (-1)^i c_i \cdot (k)_i \cdot \chi(N_{V-2i-2}, 2, k-i).$$

Then from (1), we see that if we write $\chi(G, 2, k)$ in the form (3), then $a_0 = b_0$ and $a_i = b_i + c_{i-1}$ for $i > 0$. Now, from (2) and the induction hypothesis, we have

$$\begin{aligned}
\chi(G^c, 2, k) &= \chi(G^c + e, 2, k) + k\chi(G^c - v - w, 2, k-1) \\
&= \chi((G-e)^c, 2, k) + k\chi((G-v-w)^c, 2, k-1) \\
&= \sum_{i=0}^{\lfloor V/2 \rfloor} b_i \cdot (k)_i \cdot \chi(K_{V-2i}, 2, k-i) \\
&\qquad\qquad + \sum_{i=1}^{\lfloor V/2 \rfloor} c_{i-1} \cdot (k)_i \cdot \chi(K_{V-2i}, 2, k-i) \\
&= \sum_{i=0}^{\lfloor V/2 \rfloor} a_i \cdot (k)_i \cdot \chi(K_{V-2i}, 2, k-i),
\end{aligned}$$

as desired. This completes the induction, and hence completes the proof of the theorem. □

As a corollary, we can find the expansion of $\chi(K_n, 2, k)$ in the form (3). Of course, this can be useful only for theoretical purposes since we already have the simple formula $\chi(K_n, b, k) = (k)_n$, which holds for any b.

Corollary 13. *We have*

$$\chi(K_n, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{n!}{2^i i!(n-2i)!} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i).$$

Proof. We have previously seen in (6) that

$$\chi(N_n, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i!(n-2i)!} \cdot (k)_{n-i}.$$

Hence we have

$$\begin{aligned}
\chi((K_n)^c, 2, k) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i!(n-2i)!} \cdot (k)_{n-i} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i!(n-2i)!} \cdot (k)_i \cdot (k-i)_{n-2i} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i i!(n-2i)!} \cdot (k)_i \cdot \chi(K_{n-2i}, 2, k-i).
\end{aligned}$$

Now, it is not hard to see that if different numbers a_i in a summation of the form (3) (whether or not those numbers correspond to the 2-bounded chromatic polynomial of a graph) are inserted into the summation in (14), then different polynomials result. Therefore, since by Theorem 12, the last displayed equation is consistent with having

$$\chi(K_n, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{n!}{2^i i!(n-2i)!} \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i),$$

this must actually be the representation of $\chi(K_n, 2, k)$, as desired. □

5.4 Unions of Graphs

In this section, we calculate the representation of the union $G \cup H$ of two disjoint graphs in the form (3). We see that the coefficients in the representation of $G \cup H$ are convolutions of the coefficients of G and H.

Theorem 14. *Let G and H be disjoint graphs with n vertices and m vertices respectively. Suppose that $\chi(G, 2, k)$ and $\chi(H, 2, k)$ have representations*

$$\chi(G, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i g_i \cdot (k)_i \cdot \chi(N_{n-2i}, 2, k-i) \tag{15}$$

and

$$\chi(H, 2, k) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i h_i \cdot (k)_i \cdot \chi(N_{m-2i}, 2, k-i). \tag{16}$$

Define $g_i = 0$ when i is not in the interval $0 \le i \le \lfloor n/2 \rfloor$ and $h_i = 0$ when i is not in the interval $0 \le i \le \lfloor m/2 \rfloor$. Then we have

$$\chi(G \cup H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor + \lfloor m/2 \rfloor} (-1)^i \left(\sum_{t=0}^{i} g_{i-t} h_t \right) \cdot (k)_i \cdot \chi(N_{n+m-2i}, 2, k-i).$$

Proof. By Lemma 8, we have

$$\chi(G \cup H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i g_i \cdot (k)_i \cdot \chi(N_{n-2i} \cup H, 2, k-i).$$

Similarly, as we have seen in Corollary 9, we have

$$\chi(H \cup N_i, 2, k) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j h_j \cdot (k)_j \cdot \chi(N_{m+i-2j}, 2, k-j).$$

Inserting this last expression into the formula for $\chi(G \cup H, 2, k)$ above and simplifying, we find that

$$\chi(G \cup H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^{i+j} g_i h_j \cdot (k)_{i+j} \cdot \chi(N_{n+m-2(i+j)}, 2, k-(i+j)).$$

Letting $I = i + j$ and $t = j$, we have

$$\chi(G \cup H, 2, k) = \sum_{I=0}^{\lfloor n/2 \rfloor + \lfloor m/2 \rfloor} \sum_{t=0}^{I} (-1)^I g_{I-t} h_t \cdot (k)_I \cdot \chi(N_{n+m-2I}, 2, k-I),$$

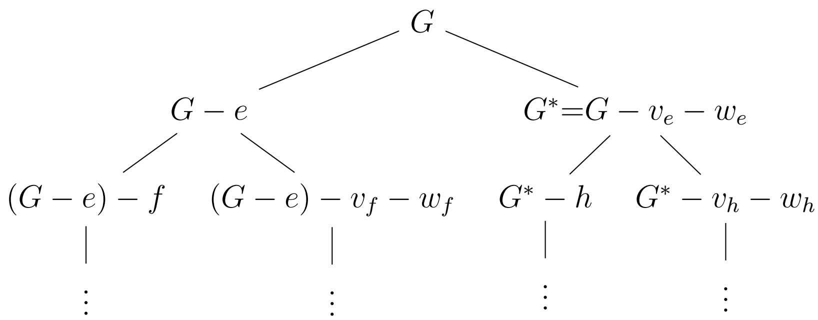
which is equivalent to the expression in the statement of the theorem. This completes the proof. □

Note that in our final expression above, the upper limit on I is $\lfloor n/2 \rfloor + \lfloor m/2 \rfloor$, whereas the form (3) indicates that the upper limit should be $\lfloor (n+m)/2 \rfloor$. If it happens that $\lfloor (n+m)/2 \rfloor > \lfloor n/2 \rfloor + \lfloor m/2 \rfloor$, then this simply tells us that $a_{\lfloor (m+n)/2 \rfloor} = 0$.

5.5 Combinatorial Interpretation of the Coefficients

In this section, we will prove Theorem 5, giving a combinatorial interpretation of the numbers a_i in the representation (3) of $\chi(G, 2, k)$ in terms of the 2-bounded chromatic polynomials of null graphs. To prove the theorem, we must show that a_i equals the number of sets of i mutually disjoint edges in G.

Consider a sequence of uses of the relation (1) to break down $\chi(G, 2, k)$ until all the resulting 2-bounded chromatic polynomials have null graphs as arguments. We may keep track of the graphs involved in these polynomials by making a binary tree where the nodes represent graphs. At every node, moving down and to the left represents deleting an edge, and moving down and to the right represents deleting the two vertices connected by that edge. For example, if we have a graph G and delete the edge $e = \{v_e, w_e\}$, and then delete the edge $f = \{v_f, w_f\}$ from the graph $G - e$ and delete the edge $h = \{v_h, w_h\}$ from the graph $G - v_e - w_e$, then the tree will be as follows.



The number a_i counts the number of terminal nodes of the tree which are assigned the null graph N_{n-2i}. We will show that this is the same as the number of sets of i pairwise disjoint edges in G.

First, we see that in order to arrive at $\chi(N_{n-2i}, 2, k-i)$, we must perform the $G-v-w$ operation exactly i times. Since the sets of vertices removed must be mutually disjoint, we see that a_i can be at most the number of sets of i mutually disjoint edges. On the other hand, suppose that S is a set of i mutually disjoint edges in G. Then there is a path in the tree from G to a terminal node in which the $G-v-w$ operation is performed exactly on the edges in S. To see this, starting at G, simply move down the tree, following vertex deletions only when the edge being removed is in S. Since every edge is ultimately removed, such a path must exist. Thus, every set of i disjoint vertices corresponds to a node with a null graph N_{n-2i}, and so the number a_i of such nodes is at least the number of sets of i disjoint vertices. This completes the proof of the theorem.

Corollary 15. *Let G be a graph with n vertices and m edges, and write $\chi(G, 2, k)$ in the form* (3)*. Then we have*

$$\begin{aligned} a_0 &= 1 \\ a_1 &= m \\ a_2 &= \frac{m(m+1)}{2} - \frac{1}{2}\sum_{v \in G} \deg(v)^2, \end{aligned}$$

where the final sum is over the vertices of G.

Proof. The first two statements are clear from Theorem 5. There is exactly one way to chose a set of 0 disjoint edges from G, and exactly m ways to choose a set of exactly one disjoint edge.

For a_2, we need to calculate the number of pairs of disjoint edges in G. To do this, let E be the set of edges in G. We first focus on a single edge $e = \{v, w\}$. Note that there are $\deg(v) + \deg(w) - 1$ edges of G (including e itself) which are not disjoint to e, and hence there are $m+1-\deg(v)-\deg(w)$ edges that are disjoint to e. Summing this expression over all edges counts every pair of disjoint edges exactly twice, so we find that

$$\begin{aligned} a_2 &= \frac{1}{2} \sum_{\{v,w\} \in E} (m + 1 - \deg(v) - \deg(w)) \\ &= \frac{m(m+1)}{2} - \frac{1}{2} \sum_{\{v,w\} \in E} (\deg(v) + \deg(w)) . \end{aligned}$$

For each vertex v, the number $\deg(v)$ appears in the above summation exactly $\deg(v)$ times. Hence, we arrive at

$$a_2 = \frac{m(m+1)}{2} - \frac{1}{2} \sum_{v \in G} \deg(v)^2,$$

as desired. This completes the proof of the corollary. □

5.6 Complete Bipartite Graphs

As an application of Theorem 5, we prove the following formula for the 2-bounded chromatic polynomial of a complete bipartite graph.

Theorem 16. *If $K_{n,m}$ is a complete bipartite graph, then we have*

$$\chi(K_{n,m}, 2, k) = \sum_{i=0}^{\lfloor (n+m)/2 \rfloor} (-1)^i i! \binom{n}{i} \binom{m}{i} \cdot (k)_i \cdot \chi(N_{n+m-2i}, 2, k - i).$$

Proof. Consider $K_{n,m}$ as the join $N_n \nabla N_m$. From Theorem 5, we know that the term a_i in (3) counts the number of sets of i mutually disjoint edges in $K_{n,m}$. To count these edges, we begin by choosing i vertices $v_1, \ldots, v_i$ from N_n and i vertices $w_1, \ldots, w_i$ from N_m to be the endpoints of these edges. There are $\binom{n}{i}\binom{m}{i}$ ways to make this choice. Each vertex v_j is connected to each vertex w_k by an edge, and so the number of sets of disjoint edges with these vertex sets is exactly counted by the number of bijections $f : \{v_1, \ldots, v_i\} \to \{w_1, \ldots, w_i\}$, which is $i!$. Hence the total number of sets of i mutually disjoint edges in $K_{n,m}$ is $i!\binom{n}{i}\binom{m}{i}$, as desired. This completes the proof of the theorem. □

5.7 Joins of Graphs

In this final section, we find an expression for the 2-bounded chromatic polynomial of the join of two disjoint graphs. In this expression, we will not express $\chi(G\nabla H, 2, k)$ in terms of 2-chromatic polynomials of null graphs, but rather in terms of 2-chromatic polynomials of complete bipartite graphs. Since Theorem 16 gives an expression for $\chi(K_{n,m}, 2, k)$, Theorem 17 below allows us to find a formula for $\chi(G\nabla H, 2, k)$ for any graph $G\nabla H$.

Theorem 17. *Let G and H be disjoint graphs with n vertices and m vertices respectively, and suppose that $\chi(G, 2, k)$ and $\chi(H, 2, k)$ are given by the expressions* (15) *and* (16)*, respectively. Then we have*

$$\chi(G\nabla H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^{i+j} g_i h_j \cdot (k)_{i+j} \cdot \chi(K_{n-2i,m-2j}, 2, k-(i+j)).$$

Proof. To begin, note that if $e = \{v, w\}$ is an edge of G, then we have $(G\nabla H) - e = (G-e)\nabla H$ and $(G\nabla H) - v - w = (G - v - w)\nabla H$. Therefore, if we use (1) to remove all the edges of G from $G\nabla H$, then we find that

$$\chi(G\nabla H, 2, k) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i g_i \cdot (k)_i \cdot \chi(N_{n-2i}\nabla H, 2, k-i).$$

In the same way, we now break down each polynomial $\chi(N_{n-2i}\nabla H, 2, k-i)$ by removing all the edges of H. This gives us

$$\begin{aligned}
&\chi(G\nabla H, 2, k) \\
&\quad = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i g_i \cdot (k)_i \cdot \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j h_j \cdot (k-i)_j \cdot \chi(N_{n-2i}\nabla N_{m-2j}, 2, k-i-j) \\
&\quad = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^{i+j} g_i h_j \cdot (k)_{i+j} \cdot \chi(K_{n-2i,m-2j}, 2, k-(i+j)),
\end{aligned}$$

as desired. This completes the proof. □

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